The Logic of Ambiguity: The Propositional Case

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Abstract. We present a logical calculus extending the classical propositional calculus with an additional connective which has some features of substructural logic. This results in a logic which seems to be suitable for reasoning with ambiguity. We use a Gentzen style proof theory based on multi-contexts, which allow us to have two ways to combine formulas to sequences. These multi-contexts in turn allow to embed both features of classical logic as well as substructural logic, depending on connectives, which would be impossible with simple sequents. Finally, we present an algebraic semantics and a completeness theorem.

1 Introduction

The term LINGUISTIC AMBIGUITY designates cases where expressions of natural language give rise to two or more sharply distinguished meanings.[1] We let the ambiguity between two meanings m_1, m_2 be denoted by $m_1 \| m_2$. Ambiguity is usually considered to be a very heterogeneous phenomenon, and this is certainly true as far as it can arise from many different sources: from the lexicon, from syntactic derivations, semantic sources as quantifiers (this is sometimes reduced to syntax), and finally from literal versus collocational meanings. Despite this, we have recently argued (see [14]) that the best solution is to treat ambiguity consistently as part of semantics, because there are some properties which are consistently present irregardless of its source. The advantage of this unified treatment is that having all ambiguity in semantics, we can use all resources in order to resolve it and draw inferences from it (we will be more explicit below). Ambiguity is a really pervasive phenomenon in natural language, but mostly does not seem to pose any problems for speakers: in some cases, we do not even notice ambiguity, whereas in other cases, we can also perfectly reason with and draw inferences from ambiguous information:

(1) The first thing that strikes a stranger in New York is a big car.

Here for example, without any explicit reasoning, the conclusion that in New York there is at least one big car (and probably many more) seems sound to us. Hence we can easily draw inferences from ambiguous statements. This entails two things for us:

[1] This roughly distinguishes ambiguity from cases of vagueness [10].

1. We should rather not disambiguate *before* we start constructing semantics, as otherwise at least one reading remains *unavailable*, and soundness of inferences cannot be verified.
2. Hence we construct something as "ambiguous meanings", and it is perfectly possible to reason with them.

As regards 1., we have to add that disambiguating before interpreting is often plausible from a psychological point of view, and in many cases ambiguity can go completely unnoticed. However, from a logical point of view this prevents sound reasoning, and our goal here is to provide a theory for sound (and complete) reasoning with ambiguity, not a psychological theory.

We will present a logic of ambiguity which extends classical logic with an additional connective $\|$, a fusion style connective which is non-commutative (for now), non-monotonic in both directions, but which allows for both contraction and expansion. Moreover, it is self-dual, which is probably its most remarkable feature, a feature we find for example in classical bilinear logic (see [6]). Another remarkable property of our calculus is that we extend classical contexts to multi-contexts, hence embed classical logic into a larger logic. We find this idea briefly mentioned in [12], but to our knowledge this seems to be the first place where it is explicitly spelled out (though the idea of using multi-contexts is not new, see [2],[5]).

The paper is structured as follows: we firstly introduce the key properties of ambiguity, in particular in relation to logic. Then we present the logic AL, an algebraic semantics, and establish its soundness and completeness.

2 Logic and ambiguity

2.1 Background and motivation

From a philosophical point of view, one often considers ambiguity to be a kind of "nemesis" of logical reasoning; for Frege for example, the main reason to introduce his logical calculus was that it was in fact unambiguous, contrary to natural language (but the discussion about the detrimental effect of ambiguity in philosophy can be traced back even to the ancient world, see [13], and is still going on, see [1]. On the other hand, in natural language semantics, there is a long tradition of dealing both with ambiguity and logic, since if we translate a natural language utterance into an unambiguous formal language such as predicate logic, ambiguity does not go away, but becomes visible by the fact that there are several translations. To consider a famous example:[2]

(2) `Every boy loves a movie.`

(3) $\exists x.\forall y.movie(x) \wedge (boy(y) \rightarrow loves(y, x))$

(4) $\forall y.\exists x.movie(x) \wedge (boy(y) \rightarrow loves(y, x))$

[2] Technically, this translation presupposes the existence of a boy, this however is irrelevant to our argument.

So we cannot simply translate natural language into logical representations (predicate logic or other), as there is no way to represent ambiguity in these languages. Note, by the way, that if we would use disjunction of (3) and (4), the formula would be logically equivalent to (3), hence this is no viable option. The standard way around the lack of functional interpretation is that we do not interpret *strings*, but rather *derivations*: one string has several syntactic derivations, and derivations in turn are functionally mapped to semantic representations (e.g. see [8]). The problem with this approach is that we basically ban ambiguity from semantics: we first make an (informed or arbitrary) choice, and then we construct an unambiguous semantics. Now this is a problem, as we have seen above:

1. If we simply pick one reading, we cannot know whether a conclusion is generally valid or not, because we necessarily discard some information.
2. To decide on a reading, we usually use semantic information; but if we choose a reading before constructing a semantic representation, how are we suppose to decide?

Now these two reasons indicate that we should not prevent ambiguity from entering semantics, because semantics is where we need it, and if it is only to get rid of it. But once ambiguity enters into semantics, we have to reason about its combinatorial, denotational and inferential properties.

For reasons of space, we will here only briefly expose what for us are the key features of ambiguity. For more extensive treatment, we refer the reader to [14]. Our exposition briefly lays out what are the challenges in developing a logic of ambiguity, and what are the key features it should have. We also want to quickly address the main reasons why ambiguity cannot be adequately treated with disjunction, which is a long-lasting misunderstanding among many scholars, even though this has been recognized many years ago, see for example [11].

2.2 Key aspects of ambiguity

1. Universal distribution For the combinatorics of $\|$, the most prominent (though only recently focussed, see [14]) feature of ambiguity is the fact that it equally distributes over all other connectives. To see this, consider the following examples:

(5) a. `There is a bank.`
    b. `There is no bank.`

((5-a)) is ambiguous between $m_1 =$"there is a financial institute" and $m_2 =$"there is a strip of land along a river". When we negate this, the ambiguity remains, with the negated content: ((5-b)) is ambiguous between $n_1 =$"there is no financial institute" and $n_2 =$"there is no strip of land along a river", and importantly, the relation between the two meanings n_1 and n_2 is intuitively exactly the same as the one between m_1 and m_2. This distinguishes an ambiguous expression such as `bank` from a hypernym as `vehicle`, which is just more *general* than the meanings "car" and "bike":

(6) a. `There was a vehicle.`
b. `There was no vehicle.`

((6-a)) means: "there was a car *or* there was a bike *or*..."; but (6-b)) rather means: "there was no car *and* there was no bike *and*...". Hence when negated, the relation between the meanings changes from a disjunction to a conjunction (as we expect from a classical logical point of view); but for ambiguity, nothing like this happens: the relation remains invariant. This also holds for all other logical operations (see [14]). This invariance is the first point where we see a clear difference between ambiguity and disjunction. This property of *universal distribution* seems to be strongly related to another observation: we can treat ambiguity as something which happens in semantics (as we do here), or we can treat it as a "syntactic" phenomenon, where "syntactic" is to be conceived in a very broad sense. In our example, this would be to say: there is not one word (as form-meaning pair) `bank`, but rather two words $\texttt{bank}_1$ and $\texttt{bank}_2$, bearing different meanings. The same holds for genuine syntactic ambiguity: one does not assume that the sentence `I have seen a man with a telescope` has strictly speaking two meanings, one rather assumes it has two derivations (thus the string represents really two distinct sentences), where each derivation comes with a single meaning. Universal distribution is what makes sure that semantic and syntactic treatment are completely parallel: every operation f on an ambiguous meaning $m_1\|m_2$ equals an ambiguity between two (identical) operations on two distinct meanings, hence

(7) $$f(m_1\|m_2) = f(m_1)\|f(m_2)$$

Note that in cases where we combine ambiguous meanings with ambiguous meanings, this leads to an exponential growth of ambiguity, as is expected. Hence universal distribution is what creates the parallelism between semantic and syntactic treatment of ambiguity. This means: strictly speaking, we do not even need to argue whether ambiguity is a syntactic or semantic phenomenon – because the result in the end should be the same, it is of no relevance where ambiguity comes from. However, as soon as we start to *reason* with ambiguity, a unified semantic treatment will only have advantages, as all information is in one place. As we only consider propositional logic, (7) reduces to

(8) $$\neg(\alpha\|\beta) \equiv \neg\alpha\|\neg\beta$$

(9) $$(\alpha\|\beta) \vee \gamma) \equiv (\alpha \vee \gamma)\|(\beta \vee \gamma)$$

(10) $$(\alpha\|\beta) \wedge \gamma) \equiv (\alpha \wedge \gamma)\|(\beta \wedge \gamma)$$

By convention, we use symbols as m_1, m_2 if we speak about (propositional) linguistic meanings, symbols like a, b, c when we speak about arbitrary algebraic objects; Greek letters α, β etc. will be reserved for logical formulas. Logically speaking, this means that $\|$ is SELF-DUAL: $\|$ preserves over negative contexts such as negation, as fusion in [6] (this logic is however used for a very different purpose, namely syntactic analysis).

Entailments An ambiguity $m_1\|m_2$ is generally characterized by the fact the speaker intends one of m_1 or m_2. The point is: we do not know which one of the two, as for example in

(11) Give me the dough!

From this simple fact, we can already deduce that for arbitrary formulas $\phi, \alpha, \beta, \chi$ in the logic of ambiguity, if $\phi \vdash \alpha \vdash \chi$ and $\phi \vdash \beta \vdash \chi$ hold, then $\phi \vdash \alpha\|\beta \vdash \chi$ holds, hence in particular, $\alpha \wedge \beta \vdash \alpha\|\beta \vdash \alpha \vee \beta$. But: we cannot reduce $\alpha\|\beta$ to neither α nor β: we have $\alpha \nvdash \alpha\|\beta$ and $\beta \nvdash \alpha\|\beta$, and also $\alpha\|\beta \nvdash \alpha$ and $\alpha\|\beta \nvdash \beta$. This is because our logic is supposed to model which inferences are valid in *every* case (i.e. under every intention), not in *some* cases, and all the latter entailments are all invalid in some cases. Considering (11), the speaker either means "pastry" or "money", but he might complain either when you given him the money or when you give him the pastry. Hence $\|$ does not coincide with any classical connective and is not Boolean definable. It is actually a **substructural** connective (see [12] for an introduction), behaving similar as fusion in linear logic: in particular, it does not allow for weakening (we will make this precise below). Note that this also illustrates how ambiguity behaves differently from disjunction.

Conservative extension In particular in connection with logic, it should be clear that our logical calculus of ambiguity should be a conservative extension of the classical calculus. The reason is that even if we include ambiguous propositions, unambiguous propositions should behave as they used to before – if there are new entailments, they should only concern ambiguous propositions.

Monotonicity/consistency Imagine someone telling you something about `banks`, and as he goes on, you discover that what he says does not make any sense to you. In the end, you notice that he has been using the term `bank` with different meanings in different utterances. At this point, you will obviously have to consider his entire discourse meaningless: how can you possibly reconstruct what meaning was intended in which utterance? Hence reasoning with ambiguous information presupposes UNIFORM USAGE: terms with several senses must be used consistently in one sense. And in fact, arguments with ambiguous terms fail if this principle of uniform usage is violated; this marks the line between their *use* and *abuse*. Hence we have the following principle:

(UU) In a given context, an ambiguous statement must be used consistently in *only one* sense.

This is of course very arguable, not only because the notion of "context" remains vague, but also because we can use the same word with different meanings in the same sentence, as in `I spring over a spring in spring.`[3]. There is a lot to say on this issue, for us however (UU) remains a technical necessity. (UU)

[3] Thanks to an anonymous reviewer for this example!

also clarifies the following important point: whereas in classical reasoning, we have inconsistency by logical contradiction, in reasoning with ambiguity, there is another source of inconsistency, namely inconsistent usage of ambiguous terms. Logically, uniform usage has its counterpart in the following inference (we denote this by monotonicity):

$$\text{(monotonicity)}\ \frac{\alpha \vdash \gamma \quad \beta \vdash \delta}{\alpha \| \beta \vdash \gamma \| \delta}$$

In our logic, this will rule will be admissible, though we formulate rules in a more general way for technical reasons.

There are some more important properties of ambiguity, such as the one that it is not productive and hence in natural language, we do not find arbitrary ambiguities. However, as neither there seems to be an apriori restriction on ambiguity, we will ignore this issue (and some others) for the moment.

2.3 A note on the standard treatment

As we have said, in natural language semantics it is already common to represent ambiguous meanings in one way or other. The standard approach for representing ambiguity (as e.g. in the quantifier case) is to use a sort of meta-semantics,[4] whose expressions underspecify the logical representations (see for example [3]). Assume our "unambiguous" language is the logic $\mathcal{L}$; and call the meta-language $\mathcal{M}$, where for example χ is a formula of $\mathcal{M}$ underspecifying the two formulas α, β of $\mathcal{L}$ (for example (3) and (4)). But now that we have this meta-language $\mathcal{M}$ of our logic $\mathcal{L}$, there are new questions:

1. How do we interpret terms of $\mathcal{M}$?
2. How do we provide connectives of $\mathcal{M}$ with a compositional semantics?
3. What are the inferences both in $\mathcal{L}$ and $\mathcal{M}$ we can draw from terms in $\mathcal{M}$?

Once we start seriously addressing these questions, we see that moving to a meta-language does not solve any problems – at best, it removes them from our sight. We usually do have a compositional semantics and consequence relation for $\mathcal{L}$; for $\mathcal{M}$ we do not. Hence $\mathcal{M}$ fails to have the most basic features of a semantics, unless, of course, $\mathcal{M}$ itself is a logic with consequence relation and compositional semantics. But in this case, considering that $\mathcal{M}$ should conservatively extend $\mathcal{L}$, it seems to be much more reasonable to include the new operator for ambiguity into our object language $\mathcal{L}$. And this is exactly what we do here. From this example it becomes once more clear that ambiguity cannot be reasonably interpreted the same way as disjunction: because $\mathcal{L}$ in any normal case already has disjunction, and there would be no need at all for $\mathcal{M}$.

[4] Actually, this would be a meta-metalanguage, because logical representations are already a form of representation of real meanings.

3 The ambiguity logic AL

3.1 Multi-sequents and Contexts

We want to advert the reader that from the presentation of AL, it will not be immediately clear how it relates to ambiguity. This will be much more obvious for its semantics, universal distribution algebras, which we present in section 4; hence if the reader is interested in the motivation rather than the logic, we advise him to first consider section 4. The logic AL is a conservative extension of classical (propositional) logic, that is, it derives all and only the valid sequents of classical logic in the language of the latter, but it has an additional connective $\|$, with which we can derive additional valid sequents. $\|$ is not very exotic from the point of view of substructural logic: it is a fusion-style operator, which allows for contraction and expansion (its inverse), but not for weakening; we can present it both in a commutative and non-commutative version. Our approach differs from the usual approach to substructural logic in that we *extend* classical logic with a substructural connective, whereas usually, one considers logics which are proper fragments of classical logic. In order to make this possible, we have to go beyond the normal sequent calculus: we still have sequents, but we have different types of contexts: one we denote by $\natural(...)$, which basically embeds classical logic, one we denote by $\Diamond(...)$, which allows to introduce the new connective $\|$. These contexts thus differ in what kind of connectives we can introduce in them, and what kind of structural rules are allowed in them. For technical reasons, there will also be a third, negative context $\flat(...)$, which has however a less "deep" meaning. The first two contexts can be arbitrarily embedded within each other, whereas the negative context is restricted to single formulas. We refer to the symbols $\natural, \Diamond, \flat$ as **modalities** (but they do not really relate to modal logic).

We call the resulting structures **multi-contexts**, a pair $\Delta \vdash \Gamma$, with Δ, Γ multi-contexts we call a **multi-sequent**, and the calculus a **multi-sequent calculus**. We have found this idea briefly mentioned as a way to approach substructural logic in [12], and structures similar to multi-contexts are found in [2]. Our approach is particular in that we actually extend classical propositional contexts, and as AL is but one particular instance of multi-sequent logics, we think that this field definitely deserves further study.

In order to increase readability, we distinguish contexts both by the symbols $\natural, \Diamond, \flat$, and by the type of period we use to separate formulas/contexts. This will be ',' in the classical context, so $\natural(\alpha, \beta)$ is a well-formed (classical) context. Here ','corresponds to $\wedge$ left of $\vdash$, and to $\vee$ right of $\vdash$, and allows for all structural rules. In the ambiguous context, we use ';', hence $\Diamond(\alpha;\beta)$ is a well-formed (ambiguous) context. ';' corresponds to $\|$, is self-dual, and allows for some structural rules such as contraction, but not for others, such as weakening or commutativity. For the negative context $\flat$, this problem will not arise, as it is strictly unary. Formulas are defined as usual, we have a set *Var* of propositional variables, and:

- if $p \in Var$, then $p \in \texttt{WFF}$;
- if $\phi, \chi \in \texttt{WFF}$, then $\phi \wedge \chi, \phi \vee \chi, \phi\|\chi, \neg\phi \in \texttt{WFF}$;
- nothing else is in $\texttt{WFF}$.

Next, we define multi-contexts; for sake of brevity, we refer to them simply as **contexts**.

1. $\natural(\epsilon)$, where ϵ is the empty sequence, is a well-formed, classical context, which we also call the empty context.
2. If $\gamma \in \texttt{WFF}$,then $\natural(\gamma)$ is a well-formed, classical context.
3. If $\gamma \in \texttt{WFF}$, then $\flat(\gamma)$ is a well-formed, negative context.
4. If $\Gamma_1, ..., \Gamma_i$ are well-formed contexts, then $\natural(\Gamma_1, ..., \Gamma_i)$ is a well-formed, classical context.
5. If Γ_1, Γ_2 are well-formed, non-empty contexts, then $\Diamond(\Gamma_1; \Gamma_2)$ is a well formed ambiguous context.

Note that $\flat$ is a strictly unary modality, whereas $\Diamond$ is strictly binary. This choice is somewhat arbitrary, but seems to be the most elegant way to prevent some technical problems. $\natural$ has no restriction in this sense. $\Gamma \vdash \Delta$ is a **well-formed multi-sequent**, if both Γ, Δ are well-formed, *classical* contexts. The calculus with all modalities is somewhat clumsy to write, so we have a number of **conventions** for increasing readability:

- We generally omit unary classical contexts; hence $\Diamond(\alpha; \beta)$ is short for $\Diamond(\natural(\alpha); \natural(\beta))$. We never omit negative contexts, which are always unary.
- In the same vein, we omit the outermost context in multi-sequents. We can do this because it always is $\natural(...)$, otherwise the sequent would not be well-formed. As a special case, we omit the empty context $\natural()$. Hence $\vdash \alpha$ is a shorthand for $\natural() \vdash \natural(\alpha)$ etc.
- We write Γ to refer to arbitrary contexts, so α, Γ is a shorthand for $\natural(\natural(\alpha), \Gamma)$;
- We write $\Gamma[\alpha]$ to refer to a subformula α of a context Γ; same for $\Gamma[\Delta]$, where Δ is a sub-context.
- We write $\Gamma[_\natural\alpha]$ etc. in order to indicate that α does occur in the scope of $\natural$, that is, the smallest sub-context containing it is classical.

We urge the reader to be careful: we will make full use of these conventions already in the presentation of the sequent calculus. The reason is that only this way, it will be plain obvious that our calculus is a nice extension of classical logic. Moreover, we aim to formulate the calculus in a way to make the structural rules of contraction and weakening admissible, as far as they are desired (see [9] for background), thoug we cannot prove these properties here for reasons of space. For the same reason, we skip the proof of basic properties such as the fact that all rules preserve well-formedness of multi-sequents, which in fact is not entirely trivial.

3.2 The classical context and its rules

The modality $\natural$ (partly) embeds the classical calculus; hence we have the following well-known rules:

(ax) $\overline{\alpha, \Gamma \vdash \alpha, \Delta}$

$$(\wedge\mathrm{I})\ \frac{\Gamma[_\natural \alpha,\beta]\vdash\Theta}{\Gamma[_\natural \alpha\wedge\beta]\vdash\Theta}\qquad(\mathrm{I}\wedge)\ \frac{\Gamma\vdash\Theta[\alpha]\quad\Gamma\vdash\Theta[\beta]}{\Gamma\vdash\Theta[\alpha\wedge\beta]}$$

$$(\vee\mathrm{I})\ \frac{\Gamma[\alpha]\vdash\Theta\quad\Gamma[\beta]\vdash\Theta}{\Gamma[\alpha\vee\beta]\vdash\Theta}\qquad(\mathrm{I}\vee)\ \frac{\Gamma\vdash\Theta[_\natural \alpha,\beta]}{\Gamma\vdash\Theta[_\natural \alpha\vee\beta]}$$

Note that in (I$\wedge$),($\vee$I) there is no requirements regarding the context. ($\wedge$I) and (I$\vee$) show how $\wedge,\vee$ correspond to ',', depending on the side of $\vdash$. The classical rules of negation introduction are *not* part of the calculus, however, they are admissible in it; this will be clear when we introduce the negative context $\flat$. In the following, we have the three structural rules of classical logic; these rules are of course bound to the classical context. We conjecture that weakening and contraction are admissible in the calculus (usual argument of reducing the degree of the rule), so the only rule we really need is commutativity.

$$(\natural\mathrm{comm})\ \frac{\Gamma[_\natural \Psi,\Theta]}{\Gamma[_\natural \Theta,\Psi]}.\qquad(\natural\mathrm{weak})\ \frac{\Gamma[_\natural \Delta]}{\Gamma[_\natural \Delta,\Psi]}\qquad(\natural\mathrm{contr})\ \frac{\Gamma[_\natural \Delta,\Delta]}{\Gamma[_\natural \Delta]}$$

This notation means that the rules can be equally applied on both sides of $\vdash$. Note that we have all these rules not for formulas, but for contexts (recall that in our notation, a formula is just a shorthand for an atomic context anyway). Finally, note that we cannot explicitly introduce $\natural$ at any point, and neither eliminate it explicitly. But our rules have a number of implicit eliminations of $\natural$, for example when we combine two formulas to one.

3.3 The ambiguous context and its rules

$\Diamond$ is a strictly binary modality, and hence there should be no way to introduce single formulas in this context. The introduction rules for $\Diamond$ are as follows:

$$(\Diamond\mathrm{I1})\ \frac{\Gamma,\Lambda\vdash\Delta,\Psi\quad\Theta,\Lambda\vdash\Phi,\Psi}{\Diamond(\Gamma;\Theta),\Lambda\vdash\Diamond(\Delta;\Phi),\Psi}\qquad(\Diamond\mathrm{I2})\ \frac{\Gamma,\Lambda\vdash\Delta\quad\Theta,\Lambda\vdash\Delta}{\Diamond(\Gamma;\Theta),\Lambda\vdash\Delta}\qquad(\Diamond\mathrm{I3})\ \frac{\Gamma\vdash\Delta,\Lambda\quad\Gamma\vdash\Phi,\Lambda}{\Gamma\vdash\Diamond(\Delta;\Phi),\Lambda}$$

Note that if we would allow the empty context in $\Diamond(;)$, then the rules ($\Diamond$I2),($\Diamond$I3) are just particular instances of ($\Diamond$I1). Hence all these rules can be seen as special instances of a single one, which would however be tedious to write down. Alternatively, we can derive ($\Diamond$I2),($\Diamond$I3) from ($\Diamond$I1) with ($\Diamond$contr). However, we rather want ($\Diamond$contr) to be admissible, as it causes an infinite search space. Consider also the particular instance of ($\Diamond$I1) where Λ,Ψ are empty: here we can see that these rules are in a sense a generalization of $\bullet$-introduction in the Lambek-calculus, and simply generalize (monotonicity) we mentioned above. There are two (parallel) introduction rules for $\|$:

$$(\|\mathrm{I})\ \frac{\Gamma[\Diamond(\alpha;\beta)]\vdash\Theta}{\Gamma[\alpha\|\beta]\vdash\Theta}\qquad(\mathrm{I}\|)\ \frac{\Gamma\vdash\Theta[\Diamond(\alpha;\beta)]}{\Gamma\vdash\Theta[\alpha\|\beta]}$$

At the same time, these rules eliminate the $\Diamond$-context. There are two structural rules in $\Diamond$-context, namely associativity and contraction (we do for now not allow commutativity), where for the latter we conjecture admissibility.

$$(\Diamond\text{ass})\ \frac{\Psi[\Diamond(\Gamma;\Diamond(\Delta;\Theta))]}{\Psi[\Diamond(\Diamond(\Gamma;\Delta);\Theta))]} \qquad (\Diamond\text{contr})\ \frac{\Gamma[\Diamond(\alpha;\alpha)]}{\Gamma[\Diamond(\alpha)]}$$

Here double lines indicate that the rule works in both directions, and absence of $\vdash$ means rules work equally on both sides. We still need a rule which ensures that we will always satisfy the distributive laws as required. The introduction rules for $\Diamond$ are already sufficient to derive $\wedge, \vee$-distribution over $\|$ *in the atomic case*; however, they interact with the negation rules (see below) in a manner not sufficiently strong to ensure that this holds for $\dashv\vdash$ as a *congruence* as long as we do not use cut. Therefore, we include the following unproblematic rule:

$$(\text{distr})\ \frac{\Gamma[\natural\Diamond(\Delta;\Psi),\Theta_1] \quad \Gamma[\natural\Diamond(\Delta;\Psi),\Theta_2]}{\Gamma[\Diamond(\natural(\Delta,\Theta_1);\natural(\Psi,\Theta_2))]}$$

Θ_1, Θ_2 are distinct, as otherwise, we have a form of expansion, which is problematic for cut elimination. This rule slightly generalize normal distribution: if $\Theta_1 = \Theta_2$, we get simple distribution, and in this special case, the rule is also *invertible*, that is, its inversion is admissible in the calculus. It is not difficult to show that without (distr), we cannot eliminate cut.

3.4 The negative context

$\flat$ marks the negative context. We design it in a way such that it subsumes classical negation rules. For this reason, we have to make sure no (classical) structural rules are applied in this context: in particular, using weakening in $\flat()$ would lead us into trouble, as the meaning of ','changes with position with respect to $\vdash$. This is why $\flat$ only applies to formulas. We need this context to derive the distributional laws for $\|$ and negation, which are not derivable so far.

$$(\flat\text{I})\ \frac{\Gamma[\natural(\alpha)]}{\Gamma[\flat(\neg\alpha)]} \qquad (\flat\text{distr})\ \frac{\Gamma[\Diamond(\flat(\phi);\flat(\chi))]}{\Gamma[\flat(\phi\|\chi)]}$$

Again, these rules operate equally on both sides of $\vdash$. Note that this is the only occasion where we explicitly write $\natural(\phi)$, because in this case, the classical modality is actually cancelled and replaced by $\flat$. Note also that in ($\flat$distr), we introduce $\|$ and delete an ambiguous context.

$$(\flat\text{E})\ \frac{\Gamma,\flat(\alpha)\vdash\Delta}{\Gamma\vdash\Delta,\alpha} \qquad (\text{E}\flat)\ \frac{\Gamma\vdash\Delta,\flat(\alpha)}{\Gamma,\alpha\vdash\Delta}$$

Hence the modality is simply eliminated by changing the position. It is easy to see that this subsumes classical negation rules: it splits one step into two, thereby allowing for negation distribution over $\|$ as an intermediate step. The best way to think of $\flat$ is maybe to assume that every formula has an atomic polarity attached, which is either positive in the case of $\natural$, or negative in the case of $\flat$.

3.5 Cut Rules

We now present the cut rule. Its adaption to our multi-sequents is not entirely straightforward, as we have to be sensitive to different contexts: cut needs to be aware of the modality of the cut formula (the formula being substituted), because otherwise we might insert positive contexts into negative contexts, which would be unsound. As $\Diamond$ is a strictly binary modality, it does not play a role for cut, as it is never the modality attached to a cut formula.

$$(\natural\text{cut})\ \frac{\Gamma[\natural(\alpha)] \vdash \Psi \quad \Delta \vdash \natural(\alpha), \Theta}{\Gamma[\Delta] \vdash \Psi, \Theta} \qquad (\Diamond\text{cut})\ \frac{\Gamma[\flat(\alpha)] \vdash \Psi \quad \Delta \vdash \flat(\alpha), \Theta}{\Gamma[\Delta] \vdash \Psi, \Theta}$$

The two rules could be obviously merged into one, if we used a meta-variable for $\natural, \flat$; this would however not really simplify things. These cut rules ensure transitivity and congruence without any special cases to consider. Importantly, as every context has a particular modality, also the sequence inserted by cut comes with a modality – but it need not be the same as the one of the cut-formula!

We define the notion of a derivation as usual as a proof-tree with the leaves being the instances of (ax); a multi-sequent $\Gamma \vdash \Delta$ is derivable if it is the root of such a proof-tree. In this case, we write $\Vdash_{\mathsf{AL}} \Gamma \vdash \Delta$, meaning the sequent is derivable in AL.

4 Semantics of AL

4.1 Universal distribution algebras

We now introduce a class of algebraic models for AL. For reasons of space, we cannot dwell on algebraic properties of this class, though they are also quite useful for understanding AL. We call this class **universal distribution algebras** or **UDA**. From the axioms, it will be easy to see that it is also a nice model for ambiguity. A universal distribution algebra is an algebra $\mathbf{U} = (U, \wedge, \vee, \sim, \|, 0, 1)$, where $(U, \wedge, \vee, \sim, 0, 1)$ is a **Boolean algebra** (for background on Boolean algebras see [4],[7]), and $\|$ is a binary function satisfying the following axioms ($a \leq b$ is an abbreviation for $a \wedge b = a$ or equivalently $a \vee b = b$):

$$\begin{array}{rl}
(\|1) & (a\|b) \wedge c = (a \wedge c)\|(b \wedge c) \\
(\|2) & \sim(a\|b) = \sim a\|\sim b \\
(\text{ass}) & (a\|b)\|c = a\|(b\|c) \\
(\text{inf}) & a \wedge b \leq a\|b \leq a \vee b \\
(\text{mon}) & a\|b \leq (a \vee c)\|(b \vee d)
\end{array}$$

($\|1$) and ($\|2$) make sure $\|$ has the property of universal distribution ($\vee$ is redundant). (ass) is clear; (inf) regulates the relation '$\leq$' between ambiguous and non-ambiguous objects; (mon) the relation '$\leq$' between ambiguous objects. Note that (inf) is partly redundant, as $a\|b \leq a \vee b$ entails $a \wedge b \leq a\|b$ and vice versa (use complementation). Spelling out (mon), we can see it is basically a sort of distributive law. It is also easy to see that (mon) is equivalent to monotonicity: it states that if $a \leq a'$, $b \leq b'$, then $a\|b \leq a'\|b'$. The formulation we chose shows that **UDA** is a variety. Note that $\|$ does not have a unit element, because intuitively, there is no unit for ambiguity. In presence of distributive laws, (inf) is equivalent to (id) $a\|a = a$: (id) entails (inf), because $(a\|b) \wedge (a \wedge b) = (a \wedge b)\|(a \wedge b) = a \wedge b$, and hence by definition of $\leq$, $a \wedge b \leq a\|b$; parallel for $a \vee b$. Conversely, (inf) entails (id), because then $a = a \wedge a \leq a\|a \leq a \vee a = a$. In **UDA**, there are many interesting properties we cannot state here for reasons of space. However, from the axioms it is clear that **UDA** presents a nice model for ambiguity.

4.2 Interpretations of AL

The interpretation of AL into **UDA** is straightforward, but we have to spell it out nonetheless. We define interpretations for contexts; this is necessary for the usual inductive soundness proof. Assume $\mathbf{U} \in \mathbf{UDA}$, $\sigma : Var \to U$ is then an atomic interpretation. We define two interpretation functions $\overline{\sigma}, \underline{\sigma}$ by:

1. $\overline{\sigma}(p) = \sigma(p) = \underline{\sigma}(p)$, for $p \in Var$.
2. $\overline{\sigma}(\phi \wedge \chi) = \overline{\sigma}(\phi) \wedge \overline{\sigma}(\chi) = \underline{\sigma}(\phi \wedge \chi)$
3. $\overline{\sigma}(\phi \vee \chi) = \overline{\sigma}(\phi) \vee \overline{\sigma}(\chi) = \underline{\sigma}(\phi \vee \chi)$
4. $\overline{\sigma}(\neg\chi) = {\sim}\overline{\sigma}(\chi) = \underline{\sigma}(\neg\chi)$
5. $\overline{\sigma}(\phi\|\chi) = \overline{\sigma}(\phi)\|\overline{\sigma}(\chi) = \underline{\sigma}(\phi\|\chi)$
6. $\overline{\sigma}(\natural(\Gamma_1, ..., \Gamma_i)) = \overline{\sigma}(\Gamma_1) \vee ... \vee \overline{\sigma}(\Gamma_i)$
7. $\underline{\sigma}(\natural(\Gamma_1, ..., \Gamma_i)) = \underline{\sigma}(\Gamma_1) \wedge ... \wedge \underline{\sigma}(\Gamma)$
8. $\overline{\sigma}(\flat(\phi)) = {\sim}(\overline{\sigma}(\phi)) = \underline{\sigma}(\flat(\phi))$
9. $\overline{\sigma}(\Diamond(\Gamma; \Delta) = \overline{\sigma}(\Gamma)\|\overline{\sigma}(\Delta)$
10. $\underline{\sigma}(\Diamond(\Gamma; \Delta)) = \underline{\sigma}(\Gamma)\|\underline{\sigma}(\Delta)$

As is easy to see, $\overline{\sigma}$ and $\underline{\sigma}$ coincide on formulas, and hence in the formula case there is no reason to distinguish them. They also coincide in their interpretation of ';', but as there might be a classical context embedded, it is important to distinguish them. With $\flat$, there is no need to keep them distinct, as this context only embeds formulas.

We define truth in a model as usual: $\mathbf{U}, \sigma \models \Gamma \vdash \Delta$ iff $\underline{\sigma}(\Gamma) \leq_U \overline{\sigma}(\Delta)$; as a special case, we have $\mathbf{U}, \sigma \models \vdash \Delta$ iff $1_U \leq_U \overline{\sigma}(\Delta)$ and $\mathbf{U}, \sigma \models \Delta \vdash$ iff $\underline{\sigma}(\Delta) \leq_U 0_U$. Moreover, we define the notion of validity as usual by $\mathbf{UDA} \models \Gamma \vdash \Delta$ (stating that $\Gamma \vdash \Delta$ is valid) iff for all $\mathbf{U} \in \mathbf{UDA}$, $\sigma : Var \to U$, we have $\mathbf{U}, \sigma \models \Gamma \vdash \Delta$. We now prove soundness and completeness of **UDA**-semantics for AL, that is, $\mathbf{UDA} \models \Gamma \vdash \Delta$ iff $\Vdash_{\mathsf{AL}} \Gamma \vdash \Delta$. We start with a section on soundness.

4.3 Soundness for AL

In this section, we only prove the following lemma:

Lemma 1 *(Soundness) If* $\Vdash_{\mathsf{AL}} \Gamma \vdash \Delta$*, then* **UDA** $\models \Gamma \vdash \Delta$.

Proof. We make the usual induction over proof rules, showing they preserve correctness. We omit this for the classical rules for which the standard proofs can be taken over with minor modifications.
▶ ($\Diamond$I1) Assume $\Gamma, \Lambda \vdash \Delta, \Psi$ and $\Theta, \Lambda \vdash \Phi, \Psi$ are true in a model. Then by (mon) $\Diamond(\natural(\Gamma, \Lambda); \natural(\Theta, \Lambda)) \vdash \Diamond(\natural(\Delta, \Psi); \natural(\Phi, \Psi))$ is true, too. It is now easy to check that by distributive laws,

$$\overline{\sigma}(\Diamond(\natural(\Gamma, \Lambda); \natural(\Theta, \Lambda))) = \overline{\sigma}(\natural(\Diamond(\Gamma; \Theta), \Lambda))$$
$$\overline{\sigma}(\Diamond(\natural(\Delta, \Psi); \natural(\Phi, \Psi))) = \overline{\sigma}(\Diamond(\Delta; \Phi), \Psi)$$

Same for $\underline{\sigma}$.

▶ ($\Diamond$I2),($\Diamond$I3) are just particular instances of ($\Diamond$I1), provided we use ($\Diamond$contr), which is sound by idempotence (which in turn is equivalent to (inf)).

▶ ($\|$I),(I$\|$),(ass): the former are sound, because antecedent and consequent have actually the same interpretation; the latter is obvious.

▶ ($\flat$I) is sound because the law of double negation holds in **UDA**: hence $\overline{\sigma}(\flat(\neg\phi)) = \overline{\sigma}(\phi)$, same for $\underline{\sigma}$, hence the claim follows easily.

▶ ($\flat$distr) is sound because of ($\|$2), negation distribution.

▶ ($\flat$E),(E$\flat$) As the function $\sim$ is a bijection in all Boolean algebras, soundness of these rules (eliminating a $\sim$) is equivalent to the soundness of the classical negation introduction rules (technically, it is their contraposition).

▶ ($\vee$I) As contexts on the left of $\vdash$ are interpreted as terms over $\|, \wedge, \sim$, we show that

(#) $\quad a\|(b \vee b')\|c \leq (a\|b\|c) \vee (a\|b'\|c)$
(+) $\quad a \wedge (b \vee b') \wedge c \leq (a \wedge b \wedge c) \vee (a \wedge b' \wedge c)$
(*) $\quad {\sim}(a \vee b) \leq {\sim}a \vee {\sim}b$

from which the soundness of the rule follows by an easy induction on the complexity of the context. (+) and (*) are obvious and well-known; we prove

$$\begin{aligned}(a\|b\|c) \vee (a\|b'\|c) &= (a \vee (a\|b'\|c))\|(b \vee (a\|b'\|c))\|(c \vee (a\|b'\|c))\\ &\geq a\|(b \vee (a\|b'\|c))\|c \text{ (by (mon))}\\ &= a\|(b \vee a)\|(b \vee b')\|(b \vee c))\|c \text{ (by (}\|1\text{))}\\ &\geq a\|a\|(b \vee b')\|c\|c \text{ (by (mon))}\\ &\geq a\|(b \vee b')\|c \text{ (by (id))}\end{aligned}$$

▶ (I$\wedge$) A parallel argument to ($\vee$I): invert $\geq$ and $\leq$, and show that $(a\|b\|c) \wedge (a\|b'\|c) \leq a\|(b \wedge b')\|c$. Then we can perform the same induction on contexts.

▶ (distr) We just consider the case on the left of $\vdash$; the other case is parallel. So assume $\Gamma[(\Delta; \Psi), \Theta_1] \vdash \Xi$ and $\Gamma[(\Delta; \Psi), \Theta_2] \vdash \Xi$ are true in a model. Assume moreover that θ_1, θ_2 are formulas such that $\underline{\sigma}(\theta_1) = \underline{\sigma}(\Theta_1)$ and $\underline{\sigma}(\theta_2) = \underline{\sigma}(\Theta_2)$, which obviously exist. We can then see (because of soundness of $\vee$-rules) that $\Gamma[(\Delta; \Psi), \theta_1 \vee \theta_2] \vdash \Xi$ is true, and by distributive laws, $\Gamma[(\natural(\Delta, \theta_1 \vee \theta_2); \Psi, \natural(\theta_1 \vee \theta_2))] \vdash \Xi$ is also true. Now as $\underline{\sigma}(\Theta_1) = \underline{\sigma}(\theta_1) \leq \underline{\sigma}(\theta_1 \vee \theta_2)$, same for θ_2, it follows that $\Gamma[(\natural(\Delta, \Theta_1); \natural(\Psi, \Theta_2))] \vdash \Xi$ is also true. For the right side of $\vdash$, we just use $\wedge$ instead of $\vee$, $\overline{\sigma}$ instead of $\underline{\sigma}$.

► ($\natural$cut) We use the well-known fact that in Boolean algebras, we have $a \wedge \neg b \leq c$ iff $a \leq c \vee b$. Assume both $\Gamma[\natural\alpha] \vdash \Psi$ and $\Delta \vdash \natural\alpha, \Theta$ are true in a model, and let $\theta \in \mathtt{WFF}$ be a formula such that $\overline{\sigma}(\theta) = \overline{\sigma}(\Theta)$. Then $\Delta, \neg\theta \vdash \natural\alpha$ is true, and by congruence, so is $\Gamma[\natural(\Delta, \neg\theta)] \vdash \Psi$. Now we make an intermediate step:

$\underline{\sigma}(\Gamma[\Delta], \Theta) \leq \underline{\sigma}(\Gamma[\natural(\Delta, \Theta)])$. This can be shown by an easy induction over Γ, the crucial step being that $(a \| b) \wedge c = (a \wedge c) \| (b \wedge c) \leq (a \| (b \wedge c))$ etc.

So $\Gamma[\Delta], \neg\theta \vdash \Psi$ remains true, and by double negation elimination, so is $\Gamma[\Delta] \vdash \Psi, \theta$, where θ can be again replaced by Θ.

► ($\flat$cut) can be, as far as semantics is concerned, be conceived of a special case of ($\natural$cut), where $\alpha = \neg\alpha'$. But this reduction obviously only works on the semantic side, syntactically, the rule has to be kept distinct. □

4.4 Completeness for AL

We now present a standard algebraic completeness proof for AL and **UDA** via the Lindenbaum algebra for AL, denoted by **Linda**. Its carrier set M is the set of AL-formulas modulo logical equivalence: we write $\alpha \dashv\vdash \beta$ iff $\Vdash_{\mathsf{AL}} \alpha \vdash \beta$, $\Vdash_{\mathsf{AL}} \beta \vdash \alpha$. This relation is symmetric by definition, reflexive and transitive (by cut). We put $\alpha_{\dashv\vdash} = \{\beta : \beta \dashv\vdash \alpha\}$, and $M = \{\alpha_{\dashv\vdash} : \alpha \in \mathtt{WFF}\}$. The next step will be to show that $\dashv\vdash$, more than an equivalence relation, is a *congruence* over connectives.

Lemma 2 *Assume $\alpha_1 \dashv\vdash \beta_1$, $\alpha_2 \dashv\vdash \beta_2$. Then for $\star \in \{\wedge, \vee, \|\}$, $\alpha_1 \star \alpha_2 \dashv\vdash \beta_1 \star \beta_2$, and $\neg\alpha_1 \dashv\vdash \neg\beta_1$.*

Proof. By cases; for all classical connectives, just use standard proof; for $\|$, this is no less straightforward. □

Hence we can use the equivalence classes irrespective of representatives and define, for $m, n \in M$:

- $m \wedge n = (\alpha \wedge \beta)_{\dashv\vdash}$, where $\alpha \in m, \beta \in n$
- $m \vee n = (\alpha \vee \beta)_{\dashv\vdash}$, where $\alpha \in m, \beta \in n$
- $m \| n = (\alpha \| \beta)_{\dashv\vdash}$, where $\alpha \in m, \beta \in n$
- $\sim m = (\neg\alpha)_{\dashv\vdash}$, where $\alpha \in m$
- $1 = (p \vee \neg p)_{\dashv\vdash}$, where $p \in \mathit{Var}$
- $0 = (p \wedge \neg p)_{\dashv\vdash}$, where $p \in \mathit{Var}$

Since our calculus subsumes the classical propositional calculus, the algebra $(M, \wedge, \vee, \sim, 0, 1)$ is a Boolean algebra, where the relation $\leq$ coincides with $\vdash$ (modulo equivalence). We prove it is a universal distribution algebra:

Lemma 3 $(M, \wedge, \vee, \sim, \|, 0, 1)$ *is a universal distribution algebra.*

Proof. As $\vdash$ corresponds to $\leq$, $=$ corresponds to $\dashv\vdash$. Hence equalities fall into two subclaims, which we sometimes treat separately.
($\|$1) i. $(a \| b) \wedge c \leq (a \wedge c) \| (b \wedge c)$.

$$
\dfrac{
  \dfrac{
    \dfrac{a,c\vdash a \quad b,c\vdash c}{\Diamond(a;b),c\vdash\Diamond(a;c)}\,(\Diamond I1)
    \quad
    \dfrac{\Diamond(a;b),c\vdash c \quad \Diamond(a;b),c\vdash c}{\Diamond(a;b),c\vdash\Diamond(c;c)}\,(\Diamond I3)
  }{\Diamond(a;b),c\vdash\Diamond(a\wedge c;c)}\,(I\wedge)
  \quad
  \dfrac{
    \Diamond(a;b),c\vdash\Diamond(a;b)
    \quad
    \dfrac{a,c\vdash c \quad b,c\vdash b}{\Diamond(a;b),c\vdash\Diamond(c;b)}\,(\Diamond I1)
  }{\Diamond(a;b),c\vdash\Diamond(a\wedge c;b)}\,(I\wedge)
}{
  \dfrac{\Diamond(a;b),c\vdash\Diamond((a\wedge c);(b\wedge c))}{\dfrac{\vdots}{(a\|b)\wedge c\vdash(a\wedge c)\|(b\wedge c)}}
}\,(I\wedge)
$$

ii. $(a\wedge c)\|(b\wedge c)\leq(a\|b)\wedge c$.

$$
\dfrac{
  \dfrac{a,c\vdash c \quad b,c\vdash c}{\Diamond(\natural(a,c);\natural(b,c))\vdash c}\,(\Diamond I2)
  \quad
  \dfrac{\dfrac{a,c\vdash a \quad b,c\vdash b}{\Diamond(\natural(a,c);\natural(b,c))\vdash\Diamond(a;b)}\,(\Diamond I1)}{\Diamond(\natural(a,c);\natural(b,c))\vdash a\|b}\,(I\|)
}{
  \dfrac{\Diamond(\natural(a,c);\natural(b,c))\vdash(a\|b)\wedge c}{\dfrac{\vdots}{(a\wedge c)\|(b\wedge c)\vdash(a\|b)\wedge c}}
}\,(I\wedge)
$$

($\|$ 2) i. $\neg(a\|b)\leq\neg a\|\neg b$ We slightly abbreviate the proof:

$$
\dfrac{\dfrac{\dfrac{\Diamond(a;b)\vdash a\|b}{\Diamond(\flat(\neg a);\flat(\neg b))\vdash\flat(\neg(a\|b))}}{\flat(\neg a\|\neg b)\vdash\flat(\neg(a\|b))}}{\neg(a\|b)\vdash\neg a\|\neg b}
$$

ii. $\neg a\|\neg b\leq\neg(a\|b)$ is parallel.

(ass) Straightforward.

(inf) Consider the following (abbreviated) proof for $a\wedge b\leq a\|b$:

$$
\dfrac{\dfrac{a,b\vdash a \quad a,b\vdash b}{a,b\vdash\Diamond(a;b)}\,(\Diamond I3)}{a\wedge b\vdash a\|b}
$$

$a\|b\leq a\vee b$ can be proved similarly, but already follows algebraically.

(mon) $a\|b\leq(a\vee c)\|(b\vee d)$ is easy to derive from $a\vdash a\vee c$, $b\vdash b\vee d$ and ($\|$I1).
□

So we have a completeness result, following by the standard argument: if a sequent is valid in all algebras, it is in particular valid in **Linda**, the term algebra, hence it is derivable in the calculus.

Theorem 4 **UDA** $\models\Gamma\vdash\Delta$ *if and only if* $\Vdash_{\mathsf{AL}}\Gamma\vdash\Delta$.

5 Conclusion and further work

We have presented the logic AL, which is an extension of the classical propositional calculus with an additional ambiguity operator. The main achievements have been the following: we have introduced a multi-sequent calculus which embeds both classical and a substructural logic. For reasons of space, we could not dwell on proof-theoretic properties of this logic, but we conjecture that many structural rules are admissible. What is most interesting is decidability of the calculus, a problem we did not treat in this paper. We rather focussed on the semantics of the calculus: we provided the algebraic semantics by means of universal distribution algebras, which we proved to be sound and complete.

Of course, this work is only preliminary: the exact properties of AL and **UDA** have not even been discussed. Still, we hope that this work shows on the conceptual side that the relation between logic and ambiguity is not an entirely negative one, and that we can effectively reason with ambiguous information. On the formal side, we think that multi-contexts and the extension of classical logic by substructural connectives is a very interesting field in the study of logic, which to our knowledge has yet attracted little attention.

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